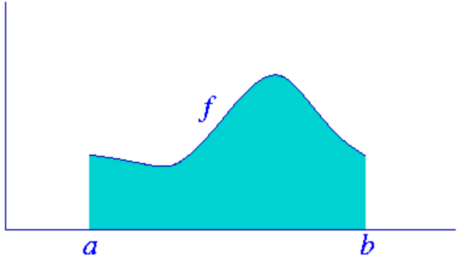
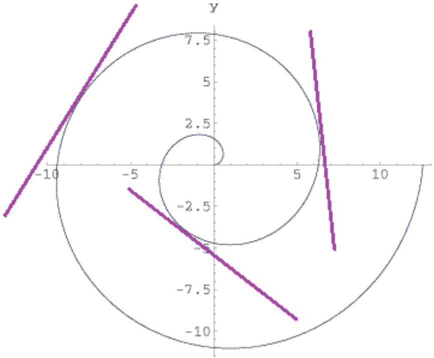
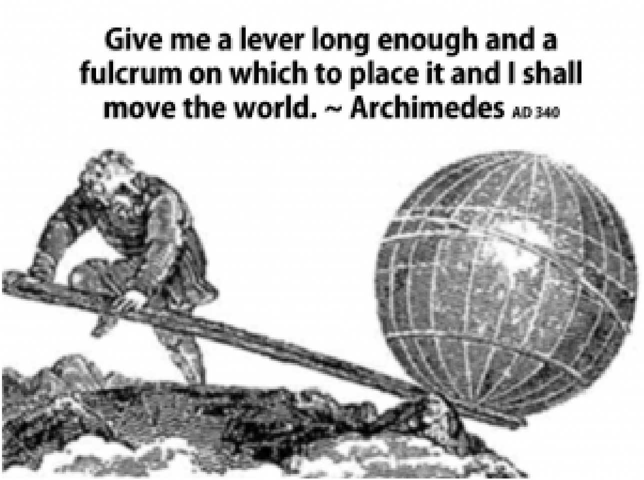
Calculus Ideas

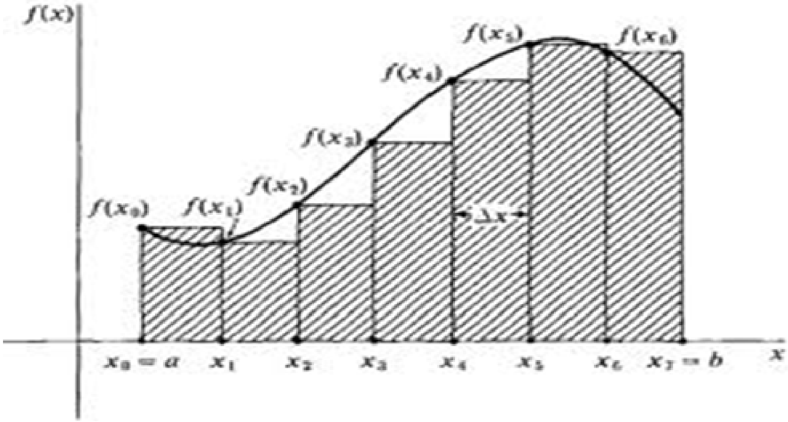




Archimedes


Give me a lever long enough and a
fulcrum on which to place it and I shall
move the world. ~ Archimedes AD 340

Riemann Sums



Riemann Integral

Definition

Let $f : [a, b] \to \mathbb{R}$ be a function and $I \in \mathbb{R}$. We say I is the Riemann integral of f, if for each $\varepsilon > 0$ there is a **number** $\delta > 0$ such that whenever $n \in \mathbb{N}$, $t_0, t_1, \cdots, t_n$ and $s_1, s_2, \cdots, s_n$ are numbers satisfying

$$a = t_0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq t_{n-1} \leq s_n \leq t_n = b$$

and

$$t_i - t_{i-1} < \delta$$

for all $i = 0, 1, \cdots n$. Then

$$|I - \sum_{i=1}^{n} f(s_i)(t_i - t_{i-1})| < \varepsilon$$

Henstock-Kurzweil Integral

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and

$$t_i - t_{i-1} < \delta(s_i)$$

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- Good convergence theorems
- It is possible to describe the Lebesgue integral without regards to measure theory

Riemann Integral Numerically

$$A(i,j) = \begin{cases} \frac{1}{i} & j \le i \\ 0 & j > i \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 \end{bmatrix}$$

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$$\lim_{n\to\infty} \sum_{j=2}^{n} (f(U(n,j) - f(U(n,j-1))\frac{1}{N}$$

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$$A(i,j) = \begin{cases} \frac{2j}{i(i+1)} & j \leq i \\ 0 & j > i \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} & 0 & 0 \\ \frac{1}{15} & \frac{2}{15} & \frac{1}{5} & \frac{4}{15} & \frac{1}{3} & 0 \\ \frac{1}{21} & \frac{2}{21} & \frac{1}{7} & \frac{4}{21} & \frac{5}{21} & \frac{2}{7} \end{bmatrix}$$

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$$\lim_{n\to\infty} \sum_{j=2}^{n} (f(U(n,j) + f(U(n,j-1)) \frac{A(n,j)}{2}$$

Generalized ODEs

Integral equation based on Riemann sums.

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Definition

Let $F : [a, b] \times [a, b] \to \mathbb{R}$ be a function and $I \in \mathbb{R}$. We say I is the Kurzweil of F, if for each $\varepsilon > 0$ there is a function $\delta : [a, b] \to (0, \infty)$ such that whenever $n \in \mathbb{N}$, $t_0, t_1, \cdots, t_n$ and $\tau_1, \tau_2, \cdots, \tau_n$ are numbers satisfying

$$a = t_0 \leq \tau_1 \leq t_1 \leq \tau_2 \leq t_2 \leq \cdots \leq t_{n-1} \leq \tau_n \leq t_n = b$$

and

$$t_i - t_{i-1} < \delta(s_i)$$

for all $i = 0, 1, \cdots, n$. Then

$$|I - \sum_{i=1}^{n}[F(\tau_i, t_i) - F(\tau_i, t_{i-1})]| < \varepsilon$$

Generalized ODEs

Definition

Let $F : [a,b] \times [a,b] \to \mathbb{R}$ be a function. $x : [\alpha, \beta] \to \mathbb{R}$ is a solution of the Generalized ODE

$$\frac{dx}{d\tau} = DF(x,t)$$

on the interval $[\alpha, \beta] \subset [a,b]$ if $(x(t), t) \in [a,b] \times [a,b]$ for all $t \in [\alpha, \beta]$ and if the identity

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t)$$

holds for every $s_1, s_2 \in [\alpha, \beta]$.

Stochastic Differential Equations

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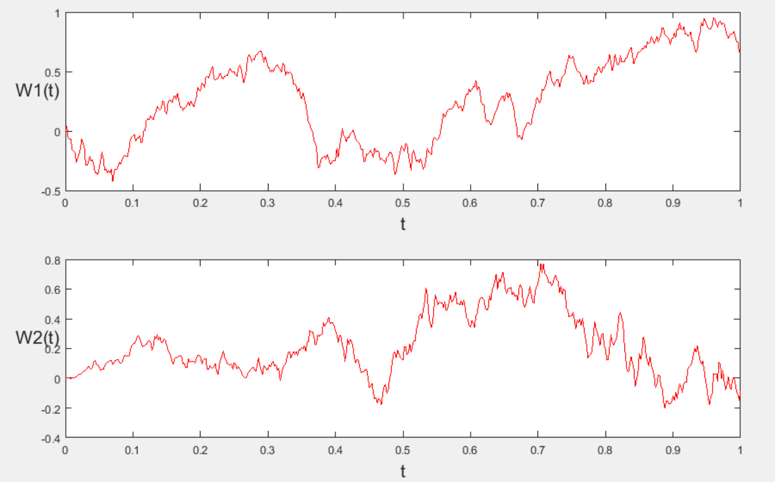

Let $h : [0,1] \to \mathbb{R}$ be a function and $W : [0,1] \to \mathbb{R}$ be a Brownian motion. Set $\delta = \frac{1}{N}$ and the partition $t_0, t_1, \cdots, t_N$, where $t_j = \frac{j}{N}$ and $N \in \mathbb{N}$.

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Ito Integral: defined by taking the limit $N \to \infty$ considering a sum of the form

$$\sum_{j=1}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

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$$\sum_{j=1}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

Stratonovich Integral: defined by taking the limit $N \to \infty$ considering a sum of the form

$$\sum_{j=1}^{N-1} h(\frac{t_j + t_{j+1}}{2})(W(t_{j+1}) - W(t_j))$$

Volterra Integral Equations

Are we able to apply HK-integral theory to Integral Equations with discontinuous solutions?

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Example

Consider the following first kind Volterra Integral Equation:

$$\int_0^t (t-s)y(s)ds = t \qquad t \in [0,1]$$

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Consider the following first kind Volterra Integral Equation:

$$\int_0^t (t-s)y(s)ds = t \qquad t \in [0,1]$$

It is known this integral equation admits the Dirac delta function as solution.

References

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