Model Reduction for Unstable Systems

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Overview

Model Reduction for Stable Linear Dynamical Systems

- Consider the linear dynamical system:

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad \text{with } \mathbf{x}(0) = \mathbf{0} \end{aligned}$$

  where $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, and $\mathbf{D} \in \mathbb{R}^{p \times m}$ are constant matrices. In this presentation we assume $m = p = 1$.
- A dynamical system is asymptotically stable or stable if all of its poles lie to the left of the imaginary axis.
- For large n e.g. $n > 10^6$, the simulation is very expensive.

Model Reduction for Linear Dynamical Systems

- Use model reduction to replace the original model with a lower dimension model:

$$\mathbf{E}_r\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{u}(t)$$
$$\mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t) + \mathbf{D}_r\mathbf{u}(t) \quad \text{with } \mathbf{x}_r(0) = \mathbf{0}$$

  where $\mathbf{A}_r, \mathbf{E}_r \in \mathbb{R}^{r\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times m}$, $\mathbf{C}_r \in \mathbb{R}^{p\times r}$, and $\mathbf{D}_r \in \mathbb{R}^{p\times m}$ with $r << n$.
- The outputs of the reduced system approximate the true outputs.
- The simulation of the reduced order model is cheaper.

Transfer Function in the Frequency Domain

- Obtain the frequency domain representation of the model by computing Laplace transforms of $\mathbf{y}(t)$, $\mathbf{y}_r(t)$ and $\mathbf{u}(t)$.
- The functions

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$
$$\mathbf{H}_r(s) = \mathbf{C}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r + \mathbf{D}_r$$

are the **transfer functions** associated with the full and reduced model respectively.

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Norms

- The $\mathcal{H}_2$ norm is defined as

$$\|\mathbf{H}(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{H}(i\omega)\|_F^2 \right)^{1/2}$$

  where $\|\cdot\|_F$ represents the Frobenius norm.
- The $\mathcal{H}_\infty$ norm is defined as

$$\|\mathbf{H}\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \|\mathbf{H}(i\omega)\|_2$$

  where $\|\cdot\|_2$ denotes the 2-norm of a matrix.

Error Measures

We aim to minimize the error between the full and the reduced model:

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$$\|\mathbf{y} - \mathbf{y}_r\|_{L_\infty} \leq \|\mathbf{H} - \mathbf{H}_r\|_{\mathcal{H}_2} \|\mathbf{u}\|_{L_2}$$

$$\|\mathbf{y} - \mathbf{y}_r\|_{L_2} \leq \|\mathbf{H} - \mathbf{H}_r\|_{\mathcal{H}_\infty} \|\mathbf{u}\|_{L_2}.$$

Rational Interpolation

- Given a set of interpolation points $\{s_i\}_i^r \subset \mathbb{C}$ we need to construct $\mathbf{H}_r$ such that

$$\mathbf{H}_r(s_i) = \mathbf{H}(s_i) \text{ for } i = 1, 2, 3, ..., r.$$

- We compute $\mathbf{H}_r$ via projection in the following manner:
  - Compute the model reduction basis $\mathbf{V}$ and $\mathbf{W}$:

$$\mathbf{V} = [(s_1\mathbf{E} - A)^{-1}\mathbf{B} \cdots (s_r\mathbf{E} - A)^{-1}\mathbf{B}]$$
$$\mathbf{W} = [\mathbf{C}(s_1\mathbf{E} - \mathbf{A})^{-1} \cdots \mathbf{C}(s_r\mathbf{E} - \mathbf{A})^{-1}]$$

  - Then, $\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\mathbf{C}_r = \mathbf{C}\mathbf{V}$, $\mathbf{B}_r = \mathbf{W}^T\mathbf{B}$.
- $\mathbf{H}_r$ satisfies Hermite interpolation conditions.

Rational Krylov Methods

We want to find $\mathbf{H}_r$ such that

$$\|\mathbf{H} - \mathbf{H}_r\|_{\mathcal{H}_2} = \min_{dim(\hat{\mathbf{H}}_r)=r} \left\|\mathbf{H} - \hat{\mathbf{H}}_r\right\|_{\mathcal{H}_2}.$$

Theorem (Gugercin and Beattie, 2014)

Let $\mathbf{H}_r(s)$ be the best r^{th} order rational approximation of $\mathbf{H}$ with respect to the $\mathcal{H}_2$ norm. Then

$$\mathbf{H}(-\lambda_k) = \mathbf{H}_r(-\lambda_k),$$
$$\mathbf{H}'(-\lambda_k) = \mathbf{H}_r'(-\lambda_k)$$

for $k = 1, 2, ..., r$ where λ_k denotes the poles of the reduced system.

Iterative Rational Krylov Algorithm (IRKA)

Sketch of IRKA

- Pick an r-fold initial shift set selection closed under conjugation
- $V = [(\sigma_1 \mathbf{E} - \mathbf{A})^{-1})\mathbf{B} \ \dots \ (\sigma_r \mathbf{E} - \mathbf{A})^{-1})\mathbf{B}]$
- $W = [(\sigma_1 \mathbf{E} - \mathbf{A})^T)^{-1}\mathbf{C}^T \ \dots \ [(\sigma_r \mathbf{E} - \mathbf{A})^T)^{-1}\mathbf{C}^T]$
- while (not converged)
  - $\mathbf{A}_r = \mathbf{W}_r^T \mathbf{A} \mathbf{V}_r, \ \mathbf{E}_r = \mathbf{W}_r^T \mathbf{E} \mathbf{V}_r, \ \mathbf{B}_r = \mathbf{W}_r^T \mathbf{B}$, and $\mathbf{C}_r = \mathbf{C} \mathbf{V}_r$
  - Compute a pole-residue expansion of $\mathbf{H}_r(\mathrm{s})$:

$$\mathbf{H}_r(s) = \mathbf{C}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r = \sum_{i=1}^{r} \frac{\phi_i}{s - \lambda_i}$$

  - $\sigma_i \leftarrow -\lambda_i$, for $i = 1, \ \dots, r$
  - $V = [(\sigma_1 \mathbf{E} - \mathbf{A})^{-1})\mathbf{B} \ \dots \ (\sigma_r \mathbf{E} - \mathbf{A})^{-1})\mathbf{B}]$
  - $W^T = [(\sigma_1 \mathbf{E} - \mathbf{A})^T)^{-1}\mathbf{C}^T \ \dots \ [(\sigma_r \mathbf{E} - \mathbf{A})^T)^{-1}\mathbf{C}^T]$
- $\mathbf{A}_r = \mathbf{W}_r^T \mathbf{A} \mathbf{V}_r, \ \mathbf{E}_r = \mathbf{W}_r^T \mathbf{E} \mathbf{V}_r, \ \mathbf{B}_r = \mathbf{W}_r^T \mathbf{B}$, and $\mathbf{C}_r = \mathbf{C} \mathbf{V}_r$

Observability and Reachability

Consider:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$
$$\Sigma := \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

- Balanced Truncation eliminates states which are hard to reach and observe.
- The reachability gramian $\mathcal{P}$ is used to classify hard to reach states.
- The observability gramian $\mathcal{Q}$ is used to classify hard to observe states.
- $\mathcal{P}$ and $\mathcal{Q}$ are the solutions of the following Lyapunov equations:

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0, \qquad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

- The values $\sigma_i = \sqrt{\lambda_i(\mathcal{P}\mathcal{Q})}$ are known as the **Hankel singular values of the system.**

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- Compute the Cholesky factor $\mathbf{U}$ of $\mathcal{P}$ and the eigendecomposition of $\mathbf{U}^T\mathcal{Q}\mathbf{U}$:

$$\mathcal{P} = \mathbf{U}\mathbf{U}^T, \quad \mathbf{U}^T\mathcal{Q}\mathbf{U} = \mathbf{K}\mathbf{G}^2\mathbf{K}^T$$

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- Compute the balancing transformation T:

$$T = \mathbf{G}^{1/2}K^T\mathbf{U}^{-1} \text{ and } T^{-1} = \mathbf{U}\mathbf{K}\mathbf{G}^{-1/2}.$$

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- The balancing state transformation yields:

$$\hat{\mathcal{P}} = \mathbf{T}\mathcal{P}\mathbf{T}^T, \; \hat{\mathcal{Q}} = \mathbf{T}^{-T}\mathcal{Q}\mathbf{T}^{-1}.$$

Truncation

- We obtain a balanced system

$$\hat{\Sigma} = \left[\begin{array}{c|c} \hat{\mathsf{A}} & \hat{\mathsf{B}} \\ \hline \hat{\mathsf{C}} & \hat{\mathsf{D}} \end{array}\right]$$

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- Consider the partitions:

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 \, \mathbf{C}_2 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & 0 \\ 0 & \mathbf{G}_{22} \end{bmatrix}$$

  for
  $\mathbf{G} = \mathrm{diag}(\sigma_1, ..., \sigma_n) = \mathcal{P} = \mathcal{Q}$, $\mathbf{G}_1 = (\sigma_1, ..., \sigma_r), \mathbf{G}_2 = (\sigma_{r+1}, ..., \sigma_N)$, where σ_i denotes the i-th Hankel singular values of the system for $i = 1, 2, ..., n$.

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- The reduced system of order r is

$$\Sigma_r = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{B}_1 \\ \hline \mathbf{C}_1 & \mathbf{D} \end{array}\right]$$

-

Model Reduction for Unstable Linear Dynamical Systems

- Consider the linear dynamical system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

- Suppose the system is unstable.
- We want to find a reduced model

$$\begin{aligned}\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r(t)\end{aligned}$$

which approximates the full order system with some error measure.

$\mathcal{L}_2$ Systems

- $\mathcal{L}_2^n(\mathbb{R}) = \{\mathbf{x}(t) \in \mathbb{R}^n | \int_{-\infty}^{\infty} \|\mathbf{x}(t)\|^2 \, dt < \infty\}$
- Magruder, Beattie and Gugercin, 2010 showed the unstable system can be associated with a bounded map from $\mathcal{L}_2(\mathbb{R})$ to itself.
- We can reduce the model with respect to the $\mathcal{L}_2$ norm defined by

$$\|\mathbf{H}\|_{\mathcal{L}_2} = (\int_{-\infty}^{\infty} |\mathbf{H}(i\omega)|^2 d\omega)^{1/2}$$

$\mathcal{L}_2$ IRKA

- Split the original systems into two systems where one is strictly stable and the other is strictly unstable.
- Negate the unstable system
- Reduce each model separately using IRKA.
- Negate the part corresponding to the unstable part.
- Combine the reduced models.
- The error is determined component wise.

System Transformation and Separation

- Let

$$\Sigma = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & 0 \end{array}\right]$$

be an unstable system with no poles in the imaginary axis.

- Suppose $\mathbf{T}$ is a transformation such that

$$\left[\begin{array}{c|c} \mathbf{TAT}^{-1} & \mathbf{TB} \\ \hline \mathbf{CT}^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} \mathbf{A}_1 & 0 & \mathbf{B}_1 \\ 0 & \mathbf{A}_2 & \mathbf{B}_2 \\ \hline \mathbf{C}_1 & \mathbf{C}_2 & 0 \end{array}\right]$$

where $\mathbf{A}_1$ is stable and $\mathbf{A}_2$ is antistable.

Observability and Reachability Gramians for Unstable Systems

- Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2 \geq 0$ be solutions to the Lyapunov equations:

$$\begin{aligned} \mathbf{A}_1\mathcal{P}_1 + \mathcal{P}_1\mathbf{A}_1^T + \mathbf{B}_1\mathbf{B}_1^T &= 0, \\ \mathbf{A}_1^T\mathcal{Q}_1 + \mathcal{Q}_1\mathbf{A}_1 + \mathbf{C}_1^T\mathbf{C}_1 &= 0, \\ (-\mathbf{A}_2)\mathcal{P}_2 + \mathcal{P}_1(-\mathbf{A}_2)^T + \mathbf{B}_2\mathbf{B}_2^T &= 0, \\ (-\mathbf{A}_2)^T\mathcal{Q}_2 + \mathcal{Q}_2(-\mathbf{A}_2) + \mathbf{C}_2^T\mathbf{C}_2 &= 0. \end{aligned}$$

- Zhou et al., 1999 showed $\mathcal{P}$ and $\mathcal{Q}$ can be computed as

$$\mathcal{P} = \mathbf{T}^{-1}\begin{bmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{bmatrix}\mathbf{T}^{-T}$$

$$\mathcal{Q} = \mathbf{T}^{-1}\begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix}\mathbf{T}^{-T}$$

Truncation

- Recall the Hankel singular values are defined as $\sigma_i = \sqrt{\lambda_i(\mathcal{P}\mathcal{Q})}$.
- $\mathcal{P} = \mathcal{Q} = diag(\sigma_1, ..., \sigma_n)$.
- Eliminate the states associated with the smallest singular values.

Reduce Unstable Systems via IRKA

- What if we reduced an unstable system by applying IRKA directly?

Reduce Unstable Systems via IRKA

- What if we reduced an unstable system by applying IRKA directly?
- While the algorithm may not converge, under certain conditions, the unstable poles are captured.
- Often, even if the poles are not captured very accurately, the stability is preserved.

Pole Capture

Initial Shifts	Final Shifts
-1.0000×10^{-1}	-9.9999×10^{-2}
-1.0000×10^{-2}	-9.9999×10^{-3}
5.5949×10^{3}	2.9678×10^{-4}
5.6076×10^{3}	1.272×10^{-2}
8.0280×10^{3}	9.9689×10^{-2}

Pole Capture

Initial Shifts	Final Shifts
-1.5321 $\times 10^{-3}$	-1.8467×10^{-2}
-1.8469 $\times 10^{-2}$	-1.0088 $\times 10^{-2} - 5.1433 \times 10^{-2} i$
-1.0069 $\times 10^{-2}$ - 5.1435 $\times 10^{-2} i$	-1.0088 $\times 10^{-2} - 5.1433 \times 10^{-2} i$
-1.0069 $\times 10^{-2}$ - 5.1435 $\times 10^{-2} i$	-1.5323 $\times 10^{-3}$
5.5737 $\times 10^{3}$	2.8033 $\times 10^{-4}$
5.5949 $\times 10^{3}$	1.2560 $\times 10^{-2}$
5.6077 $\times 10^{3}$	3.9537 $\times 10^{-2}$- 6.7733 $\times 10^{-2}$i
8.0280 $\times 10^{3}$	3.9537 $\times 10^{-2}$+ 6.7733 $\times 10^{-2}$i

Comparisons of Algorithms

r=12	$\mathcal{L}_2$ IRKA	$\mathcal{L}_2$ IRKA
Initialization	Pole reflection	Bal. truncation poles
$\mathcal{H}_2$ Relative error	0.4885	0.0457
$\mathcal{H}_\infty$ Relative error	0.5160	0.0041

r=12	IRKAfUS	IRKAfUS
Initialization	Pole reflection	Bal. truncation poles
$\mathcal{H}_2$ Relative error	0.1449	0.1754
$\mathcal{H}_\infty$ Relative error	0.0198	0.0423

r=12	Bal. Truncation	$\mathcal{L}_2$ IRKA
Initialization	-	IRKAfUS poles
$\mathcal{H}_2$ Relative error	0.0717	0.0840
$\mathcal{H}_\infty$ Relative error	0.0110	0.0072

Conclusions and Future Work

- For unstable systems with few unstable poles, numerical simulations show IRKA captures the unstable poles.
- These obtained poles appear to be good initializations for $\mathcal{L}_2$ IRKA.
- Investigate delay systems and develop techniques to reduce such systems.
- Consider second-order models and compare structure preserving algorithms with optimal ones such as IRKA.

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Thank you!