Approximation of the Linearized Boussinesq Equations

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Outline

1. Motivation
2. Iterative Rational Krylov Method (IRKA)
3. Boussinesq Problem
4. Results
5. Future Work
Motivation

- Controlling large-scale non-linear dynamical systems resulting from partial differential equation models is prohibitive and thus requires simplification, i.e. model reduction.

- One technique is to linearize the full system and then apply the control to the linear system.

- This talk focuses on reducing the linear portion of the system using the Iterative Rational Krylov Algorithm (IRKA).
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Iterative Rational Krylov Method (IRKA)

Descriptor System

- Given the system

\[ E \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^p \), \( E \) singular, \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \), and \( D \in \mathbb{R}^{p \times m} \).

- We obtain \( \hat{y}(s) = G(s)\hat{u}(s) \) where \( \hat{y}(s) \) and \( \hat{u}(s) \) are the Laplace transforms of \( y(t) \) and \( u(t) \) and the transfer function is

\[ G(s) = C(sE - A)^{-1}B + D \]

- \( G(s) \) is a rational function of degree “\( n \)”.
Our goal is to create a reduced order model of size $r \ll n$ such that

$$\tilde{E}\dot{x}_r(t) = \tilde{A}x_r(t) + \tilde{B}u(t)$$
$$y_r(t) = \tilde{C}x_r(t) + \tilde{D}u(t)$$

with $x_r(t) \in \mathbb{R}^r$, $\tilde{E}, \tilde{A} \in \mathbb{R}^{r \times r}$, $\tilde{B} \in \mathbb{R}^{r \times m}$, $\tilde{C} \in \mathbb{R}^{p \times r}$, and $\tilde{D} \in \mathbb{R}^{p \times m}$.

Similarly $\hat{y}_r(s) = \tilde{G}(s)\hat{u}(s)$ where $\hat{y}_r(s)$ and $\hat{u}(s)$ are the Laplace transforms of $y_r(t)$ and $u(t)$ and the transfer function is

$$\tilde{G}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} + \tilde{D}$$

$\tilde{G}(s)$ is a rational function of degree $r$. 

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Main Goal

- For a wide range of inputs, we seek to minimize the error between the output of the full model $y$ and the output of the reduced model $y_r$.

- This result is equivalent to minimizing the error between the transfer functions of the full model and the reduced model.

- For descriptor systems, the transfer functions may be improper rational functions. That is

$$G(s) = G_{sp}(s) + P(s)$$

$$\tilde{G}(s) = \tilde{G}_{sp}(s) + \tilde{P}(s)$$

where $P$ and $\tilde{P}$ are the polynomial parts of the transfer functions.

- In order to prevent the error from blowing up as $s \to \infty$, we must enforce equivalence of the polynomial parts, $P = \tilde{P}$.
Visual Representation of Model Reduction

- Remember that $n \gg r$.
- Although the dimension of the output and input remain the same, the full state space has been projected onto a much smaller subspace.
### Projection Framework

- In order to project the model onto the reduced state space, $n \times r$ matrices $V$ and $W$ must be constructed.

- Using $V$, the full-order state $x(t)$ is approximated by $Vx_r(t)$.

- Using $W$, the Petrov-Galerkin condition is enforced as follows

  $$W^T (EV\dot{x}_r(t) - AVx_r(t) - Bu(t)) = 0, \quad y_r(t) = CVx_r(t) + Du(t).$$

- This results in a reduced order model where

  $$\tilde{E} = W^T EV, \quad \tilde{A} = W^T AV,$$

  $$\tilde{B} = W^T B, \quad \tilde{C} = CV, \quad \tilde{D} = D.$$  

- The big question is “How do we find $V$ and $W$?”
The Optimal Reduced Order Model

- For simplicity, let us assume that we are dealing with a SISO system. That is $m = p = 1$.

- Once we solve this problem, we can easily extend to the MIMO setting where $m \neq 1$ and/or $p \neq 1$. The method for doing this is described in Gugercin, Antoulas, and Beattie [2].

- We want to find $\tilde{G}$ such that

$$\min_{\dim(\tilde{G})=r} \| G - \tilde{G} \|_{H_2} = \min_{\dim(\tilde{G})=r} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega) - \tilde{G}(i\omega)|^2 \, d\omega \right)$$

- This is equivalent to the following

$$\min \| y - \tilde{y} \|_{\infty}$$
Theorem

Given the transfer function $G(s) = C(sE - A)^{-1}B$, the optimal reduced order transfer function $	ilde{G}(s) = C(s\tilde{E} - \tilde{A})^{-1}\tilde{B}$ satisfies

\[
G(-\lambda_i) = \tilde{G}(-\lambda_i) \\
G'(-\lambda_i) = \tilde{G}'(-\lambda_i)
\]

where $\lambda_i, i = 1, \cdots, r$ are the eigenvalues of the pencil $\lambda \tilde{E} - \tilde{A}$.

Note:

- This is simply a Hermite interpolation of the rational transfer function at the points $\{-\lambda_i\}_{i=1}^r$.
- Unfortunately, we have no way of knowing the points $\{\lambda_i\}_{i=1}^r$ a priori.
Iterative Rational Krylov Method (IRKA)

- IRKA is the method we use to find the points $\{\sigma_i\}_{i=1}^r = \{-\lambda_i\}_{i=1}^r$.

- At its core, IRKA is a fixed point algorithm.

- In short, we first guess $\{\sigma_i\}_{i=1}^r$ and determine $V$ and $W$ using those points.

- While the relative change in $||\sigma|| > \text{tol}$
  1. $\tilde{E} = W^T E V$ and $\tilde{A} = W^T A V$
  2. $\sigma_i \leftarrow -\lambda_i(A_r, E_r)$ for $i = 1, \ldots, r$.
  3. Assemble new $V$ and $W$ using $\sigma_i$.

- For details see Gugercin, Stykel, and Wyatt [3].
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Natural Convection Cavity

\[ \frac{\partial T}{\partial n} = 0 \]

\[ T = T_h \quad \Omega \quad T = T_c \]

\[ \frac{\partial T}{\partial n} = 0 \]
Natural Convection Cavity

Figure: Flow after 60 seconds
Boussinesq Equation Model

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \frac{1}{Re} \Delta \mathbf{v} \\
\nabla \cdot \mathbf{v} &= 0 \\
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T &= \frac{1}{RePr} \Delta T + Bu
\end{align*}
\]

over \( \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2 \).

**Note:**
This is just incompressible Navier-Stokes coupled with the temperature convection-diffusion equation. For details see Borggaard, Burns, Surana, and Zietsman [1].
Linearized Boussinesq Equations

Linearizing the Boussinesq equations around an average velocity results in the following Stokes-type descriptor system of index 2:

\[ E_{11} \dot{x}_1(t) = A_{11} x_1(t) + A_{12} x_2(t) + B_1 u(t) \]
\[ 0 = A_{21} x_1(t) + B_2 u(t) \]
\[ y(t) = C_1 x_1(t) + C_2 x_2(t) + D u(t) \]

where the state is \( x_1 = [v, T]^T, x_2 = [p]^T \).

Notes:

1. For this model \( B_2 = 0 \) and \( D = 0 \), although this is not a requirement.
2. The control \( u(t) \) sets the temperature difference between the hot and cold walls.
3. The output \( y(t) \) is the average vorticity for the cavity.
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Results

• The full model was reduced from $n = 7301$ to $r = 15$.

• The transfer function response was compared over a wide frequency range.

• Both systems were then simulated for 10 seconds using the same control.

• Results were compared for the output and state space.
Bode Plot

Figure: System response over the frequency domain
Control Term \( u(t) \)

Figure: Temperature difference between hot and cold walls
Output $y(t)$ and $y_r(t)$ - Full and Reduced Models

![Graph showing Average Vorticity vs Time](image)

**Figure**: Average vorticity of the system
Figure: Relative error of average vorticity between full and reduced models
State Space Results

- Although the model reduction technique is designed to minimize the output error between the full and reduced models, we were curious as to how well the reduced system approximated the full state.

- By projecting the reduced model back up to the full state space using our projection matrices, we obtained the following results.
Velocity Field (Full Model) after 10 Seconds

Figure: Velocity Field
Results

Velocity Field (Reduced Model) after 10 Seconds

Final Velocity Vector Field (Reduced)

Figure: Velocity Field
State Error Between Full and Reduced Models

Figure: Error over time
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Future Work

1. Work with larger, more complex models.

2. Couple these results with POD (proper orthogonal decomposition) to reduce the full non-linear system.

3. Establish error bounds.

4. Establish controllability conditions.
Bibliography

